In the case of n-type semiconductor

From the charge neutrality condition, the electron concentration in the freeze-out range is expressed as

$$N_{\mathrm{C}}(T)\exp\left(-\frac{\Delta E_{\mathrm{F}}}{kT}\right)=N_{\mathrm{D}}\frac{1}{1+2\exp\left(\frac{\Delta E_{\mathrm{D}}-\Delta E_{\mathrm{F}}}{kT}\right)}. \tag{1}$$

When we define x as

$$x=\exp\left(-\frac{\Delta E_{\mathrm{F}}}{kT}\right), \tag{2}$$

Eq. (1) can be rewritten as a quadratic equation:

$$ax^2+bx+c=0, \tag{3}$$

where

$$a=2N_{\mathrm{C}}(T)\exp\left(\frac{\Delta E_{\mathrm{D}}}{kT}\right), \tag{4}$$

$$b=N_{\mathrm{C}}(T), \tag{5}$$

and

$$c=-N_D. \tag{6}$$

The solution of Eq. (3) is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{7}$$

or

$$x=\left(-\frac{b}{2a}\right)\pm\sqrt{\left(\frac{b}{2a}\right)^2-\left(\frac{c}{a}\right)}. \tag{8}$$

When

$$\left(\frac{b}{2a}\right)^2<<\left(\frac{c}{a}\right) \tag{9}$$

that is

$$N_{\mathrm{D}}>>\frac{N_{\mathrm{C}}(T)}{8}\exp\left(-\frac{\Delta E_{\mathrm{D}}}{kT}\right), \tag{10}$$

the solution is

$$x=\sqrt{\frac{-c}{a}} \tag{11}$$

that is

$$\exp\left(-\frac{\Delta E_{\mathrm{F}}}{kT}\right)=\sqrt{\frac{N_{\mathrm{D}}}{2N_{\mathrm{C}}(T)}\exp\left(-\frac{\Delta E_{\mathrm{D}}}{kT}\right)}. \tag{12}$$

Therefore, $n(T)$ is derived as

$$n(T) = N_{\mathrm{C}}(T)\exp\left(-\frac{\Delta E_{\mathrm{F}}}{kT}\right) = \sqrt{\frac{N_{\mathrm{D}}N_{\mathrm{C}}(T)}{2}}\exp\left(-\frac{\Delta E_{\mathrm{D}}}{2kT}\right). \tag{13}$$

Since

$$N_{\mathrm{C}}(T) = 2\left(\frac{2\pi m^{*}kT}{h^{2}}\right)^{3/2} M_{\mathrm{C}}, \tag{14}$$

we obtain the following relationship:

$$n(T) = T^{3/4}\sqrt{\left(\frac{2\pi m^{*}k}{h^{2}}\right)^{3/2} M_{\mathrm{C}}}\sqrt{N_{\mathrm{D}}}\exp\left(-\frac{\Delta E_{\mathrm{D}}}{kT}\right). \tag{15}$$